Kuwait University | **Math 101** | **Date: June 6, 2010**
Dept. of Mathematics | **Final Exam** | **Duration: 120 minutes**

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed.

Answer the following questions: Each question weighs 4 points.

Question 1.

(a) Use the definition to prove that $\lim_{x\to 6}\left(9-\frac{1}{6}x\right)=8$.

(b) Find $\lim_{x\to 2}\left(\frac{\sin(x-2)}{x^2+x-6}\right)$.

Question 2. Find all intervals on which the function $f(x)=\frac{\sqrt{9-x^2}}{x^4-16}$ is continuous.

Question 3. Let $f(x)=\frac{(x-3)^{\frac{2}{3}}}{x-1}$.

(a) Find the x-coordinate(s), if any, of the point(s) on the curve $y=f(x)$ where the tangent line is vertical.

(b) Find the x-coordinate(s), if any, of the point(s) on the curve $y=f(x)$ where the tangent line is horizontal.

Question 4. Let $f(x)=x-\frac{3}{2}x^{\frac{2}{3}}$.

(a) Find $f'(x)$.

(b) Determine whether there exists a number $c\in(-8,8)$ for which $f'(c)=0$.

(c) Which of the condition(s) of Rolle's Theorem are satisfied on $[-8,8]$? (Justify your answer)

Question 5. Find two positive numbers whose product is 100 and whose sum is a minimum.

Question 6. Evaluate

a) $\int \frac{x+1}{\sqrt{x^2+2x+7}}\,dx$ b) $\int \frac{1+\sin(x)}{\cos^2(x)}\,dx$.

Question 7. Evaluate $\int_{-1}^{1}(1-x)\sqrt{1-x^2}\,dx$.

Question 8. Find the point(s) of inflection, if any, of the curve

$$y=\int_0^x \frac{t^2-1}{t^2+1}\,dt.$$

Question 9. Find the area of the region between the curves $y=\sqrt{x}$ and $y=x$ from $x=0$ to $x=4$.

Question 10. Set up an integral for the volume of the solid obtained by rotating the region enclosed between the curves $y=x^2$ and $y=x+2$ about:

(a) $x=7$

(b) $y=-1$.

1.

(a) Use the definition to prove that $\lim_{x\to 6}\left(9-\frac{1}{6}x\right)=8$.

For any $\varepsilon>0$, we wish to find $\delta>0$ s.t. $\left|9-\frac{1}{6}x-8\right|<\varepsilon$ whenever $0<|x-6|<\delta$. $\left|9-\frac{1}{6}x-8\right|<\varepsilon \Longleftrightarrow \left|\frac{1}{6}(6-x)\right|<\varepsilon \Longleftrightarrow \frac{1}{6}|x-6|<\varepsilon \Longleftrightarrow |x-6|<6\varepsilon$. Thus, $\delta=6\varepsilon$.

(b) $\lim_{x\to 2}\left(\frac{\sin(x-2)}{x^2+x-6}\right)=\lim_{x\to 2}\left(\frac{\sin(x-2)}{(x-2)(x+3)}\right)=\frac{1}{5}$.

2. $f(x)=\frac{\sqrt{9-x^2}}{x^4-16}$ is continuous on $[-3,-2)\cup(-2,2)\cup(2,3]$.

3. Let $f(x)=\frac{(x-3)^{\frac{2}{3}}}{x-1}$. $f'(x)=\frac{\frac{2}{3}(x-3)^{\frac{-1}{3}}(x-1)-(1)(x-3)^{\frac{2}{3}}}{(x-1)^2}=\frac{2(x-1)-3(x-3)}{3(x-3)^{\frac{1}{3}}(x-1)^2}$.
Thus $f'(x)=\frac{-x+7}{3(x-3)^{\frac{1}{3}}(x-1)^2}$.

(a) The tangent line is vertical at $x=3$. Note that $x=1\notin D_f$.

(b) The tangent line is horizontal at $x=7$.

4. Let $f(x)=x-\frac{3}{2}x^{\frac{2}{3}}$.

1. $f'(x)=1-x^{\frac{-1}{3}}=\frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{3}}}$.
2. $f'(c)=0$ when $c=1$.
3. f is continuous on $[-8,8]$.

5. Let $P=x\cdot y$ and $S=x+y$, where x and y are positive real numbers. From $P=x\cdot y=100$ we derive that $y=\frac{100}{x}$. Now, S can be written as $S=x+\frac{100}{x}$. Hence, $\frac{dS}{dx}=1-\frac{100}{x^2}=\frac{x^2-100}{x^2}$. The only critical number is $x=10$. Since $f''(x)=\frac{200}{x^3}>0$ for $x=10$, we conclude that the sum is minimum when $x=10$ and $y=10$.

6.

(a) Let $u=x^2+2x+7$. Then $du=2(x+1)dx$. $\int\frac{x+1}{\sqrt{x^2+2x+7}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{u}}\,du=\sqrt{u}+C=\sqrt{x^2+2x+7}+C$.

(b) $\int\frac{1+\sin(x)}{\cos^2(x)}=\,dx=\int\left(\sec^2(x)+\sec(x)\tan(x)\right)\,dx=\tan(x)+\sec(x)+C$.

7. Evaluate $\int_{-1}^{1}(1-x)\sqrt{1-x^2}\,dx=\int_{-1}^{1}\sqrt{1-x^2}\,dx-\int_{-1}^{1}x\sqrt{1-x^2}\,dx=\frac{\pi}{2}+0=\frac{\pi}{2}$.

8. $y=\int_0^x\frac{t^2-1}{t^2+1}\,dt$. $y'=\frac{x^2-1}{x^2+1}$ and $y''=\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$. $f''(x)=0$ when $x=0$. Since $f''(x)<0$ for $x<0$ and $f''(x)>0$ for $x>0$, then the curve $y=f(x)$ has an inflection point at $x=0$. This point is $(0,0)$.

9. Points of intersection: set $x = \sqrt{x}$: $x = 0, 1$. The total area is $A = A_1 + A_2$, where:
$$A_1 = \int_0^1 \left(\sqrt{x} - x\right) dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^1 = \frac{1}{6},$$
and
$$A_2 = \int_1^4 \left(x - \sqrt{x}\right) dx = \left[\frac{x^2}{2} - \frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \frac{17}{6}.$$ Thus, $A = \frac{18}{6} = 3$.

10. Points of intersection: set $x^2 = x + 2$: $x^2 - x - 2 = 0$: $(x-2)(x+1) = 0$: $x = -1, 2$.

(a) $x = 7$: $V = \displaystyle\int_{-1}^{2} 2\pi(7 - x)[(x+2) - (x^2)]\, dx$.

(b) $y = -1$: $V = \displaystyle\int_{-1}^{2} \pi[((x+2)+1)^2 - ((x^2)+1)^2]\, dx$.